For information and correspondence:
WATER RESOURCES PUBLICATIONS
P.O. Box 2841
Littleton, Colorado 80161, U.S.A.

REPRINTED FROM THE BOOK:

RAINFALL-RUNOFF RELATIONSHIP

A PART OF THE
Proceedings of the International Symposium on Rainfall-Runoff Modeling held May 18-21, 1981 at Mississippi State University, Mississippi State, Mississippi, U.S.A.

WRP

THE MODIFIED ATTENUATION-KINEMATIC (ATT-KIN) ROUTING MODEL

George H. Comer,
Hydraulic Engineer
USDA Soil Conservation Service, National Engineering Staff
10000 Aerospace Road, Lanham, MD 20801

Fred D. Theurer,
Civil Engineer
USDA Soil Conservation Service, Cooperative Instream Flow Service Group
2625 Redwing Road, Ft. Collins, CO 80526

Harvey H. Richardson,
Hydraulic Engineer
USDA Soil Conservation Service, National Engineering Staff
10000 Aerospace Road, Lanham, MD 20801

ABSTRACT

The Soil Conservation Service (SCS) developed the modified Att-Kin routing model for valley routing of flood hydrographs. The model is simple to apply, and its accuracy is verified by comparisons with the complete solution of the unsteady flow equations. The Att-Kin model combines the best features of storage and kinematic routing models. The storage routing model gives attenuation of the routed hydrograph, but it does not give translation time. The kinematic routing model gives translation, but it results in no attenuation of the routed hydrograph. The Att-Kin (so named because it gives attenuation and kinematic translation) solves both models in a way that satisfies the conservation of mass at the time of the outflow peak. The Att-Kin model was developed theoretically from physical considerations and modified to synthesize the results from the solution of the complete unsteady flow equations.

INTRODUCTION

This paper presents the math model to be used by the Soil Conservation Service (SCS) for valley flood routing. The model, a modified Att-Kin (attenuation- kinematic) procedure, will be used in a revision of the current Computer Program for Project Formulation-Hydrology described in Technical Release 20 (TR-20) (Soil Conservation Service, 1965).

The TR-20 computer program was developed in 1964 and incorporated manual flood routing procedures, but current technology and changing requirements for the use of TR-20 called for improved yet uncomplicated procedures. The modified Att-Kin procedure fills this need.

The Att-Kin model does not alter the basic form of the original convex routing method. It provides a better physical basis for determining the routing coefficient while using a numerical model that satisfies the intended math model. The physical basis of the Att-Kin model is twofold: (1) it satisfies the time to peak of the hydrograph throughout the valley; and (2) it satisfies the conservation of mass at the time to peak. To accomplish these items, the Att-Kin model combines the best features of both the storage and the kinematic models.

DESCRIPTION OF MODELS

Before describing the convex and Att-Kin models, a brief description of the physical, mathematical, and numeric behavior of the storage and kinematic models is necessary.

The storage model is physically related to reservoir routing applications. The mathematical behavior produces instantaneous translation with maximum attenuation for the storage involved. Closed-form solutions can be developed for analytic inflow hydrographs with linear storage-discharge relationships. Simple numeric models can be constructed and easily solved that satisfy the math model without any stability problems. However, care must be taken not to needlessly subdivide the reach, because the numeric results approach a kinematic rather than storage math model solution.

The kinematic model is physically related to wave propagation situations. The mathematical behavior produces translation for any discharge with absolutely no attenuation. Simple closed-form solutions can be done for any type of inflow hydrograph and nonlinear storage-discharge relationship. Numeric models can be constructed and solved that satisfy the math model, but the reach must be divided into enough subreaches to prevent the results from converging to a storage rather than kinematic model solution.

Proper unsteady flow models describing the passage of flood hydrographs through a reach must at least be a combination of both kinematic for translation and storage for attenuation. The full dynamic equations simultaneously account for both effects, but they are very difficult to apply to general field situations. However, the coefficient models, which are simpler than unsteady flow models, can be used for most field applications if (1) a math model is chosen that describes the important physical processes, (2) the routing coefficients are determined for each application to reflect the actual site-specific physical relationships, and (3) a numeric model is constructed that satisfies the chosen math model. TR-20 currently uses a convex model that fails to meet these requirements. The Att-Kin model was developed to incorporate the strengths of the convex model while overcoming its weaknesses.

Convex Model

The current TR-20 computer program uses a valley routing equation (convex model) derived from the conservation of mass equation. The conservation of mass equation can be written as

$$I - O = dS/dt \cong \Delta S/\Delta t \quad \text{------------------------------------}(1)$$

The symbols are defined in the List of Symbols at the end of the paper.

By assuming that the inflow and outflow at the beginning of the time interval are the averages over the interval, equation 1 becomes

$$I_1 - O_1 = \Delta S/\Delta t = (S_2 - S_1)/\Delta t \quad \text{----------------------------}(2)$$

where the subscripts 1 and 2 denote the beginning and end of the time interval respectively.

Let K be the slope of the storage outflow curve. With $S = KO$, equation 2 can be rewritten as

$$O_1 = CI_1 + (1 - C)O_1 \qquad (3)$$

$$\text{where } C = \Delta t/K \qquad (4)$$

Equation 3 is the formulation of the convex method described in NEH-4 (Soil Conservation Service, 1972); however, the numerical model used to solve it has a decided tilt towards a kinematic model solution. Equation 4 is the definition for the convex routing coefficient, but the determination for K is not necessarily close to a significant point of the valley storage-discharge curve.

Both the solution of equation 4 and the numerical model used to solve equation 3 are the major weaknesses in TR-20 that the Att-Kin model will overcome.

The Att-Kin model uses a similar derivation with three exceptions: (1) an arithmetic average is assumed for the outflow, (2) the determination of the routing coefficient, C_R, is related to the site-specific physical processes, and (3) the numeric model is constructed to satisfy the math model.

Using an arithmetic outflow average in the equation for the conservation of mass gives

$$I_1 - (O_2 + O_1)/2 = (K/\Delta t)(O_2 - O_1) \qquad (5)$$

which becomes

$$O_2 = C_R I_1 = (1 - C_R)O_1 \qquad (6)$$

where

$$C_R = (2\Delta t)/(2K + \Delta t) \qquad (7)$$

The Att-Kin procedure described in TR-66 (Soil Conservation Service, 1979) was modified to provide a direct solution for estimating K. A brief discussion of the original Att-Kin model, together with the modifications, is presented in this paper. More detail can be found in TR-66.

The original Att-Kin model storage routes the inflow hydrograph with an assumed K, then kinematic routes the resulting attenuated hydrograph (see figure 1). The kinematic routed attenuated hydrograph is checked for conservation of mass between inflow, outflow, and valley storage at the time to peak of the outflow hydrograph. Appropriate corrections are made to the assumed K value and the entire process repeated until the conservation of mass at the time to peak is satisfied. This obviously requires iteration. The following paragraphs describe the Att-Kin procedure in physical-mathematical terms.

The inflow hydrograph is storage routed using the storage indication method to solve the integral form of the conservation of mass equation using a single-valued valley storage-discharge relationship. The equations are:

$$(\bar{Q}_I - \bar{Q}_o)(t_2 - t_1) = S_{o,t_2} - S_{o,t_1} \qquad (8)$$

$$Q_{o,t} = k_s S^m_{o,t} \qquad (9)$$

The attenuated hydrograph is then translated and distorted by kinematic

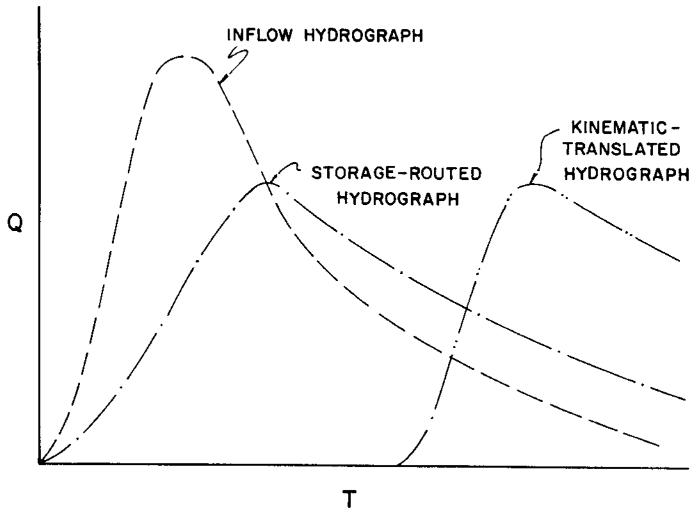

Figure 1. Hydrographs resulting from storage routing and kinematic translation.

routing. Kinematic routing solves the differential form of the conservation of mass equation using a single-valued flow area-discharge relationship. The equations used are:

$$\partial Q/\partial x + \partial Q/\partial t = 0 \quad \text{--(10)}$$

$$Q_{x,t} = k_a A_{x,t}^{m_a} \quad \text{--(11)}$$

These two routing models required two separate single-value discharge relationships. The two single-valued functions can be related by

$$S = LA \quad \text{--(12)}$$

Since $m = m_a$, the flow area discharge relationship can be determined from the valley storage-discharge relationship by

$$Q = k_a A^m = k_s L^m A^m \quad \text{--(13)}$$

Some, but not all, of the valley storage results in attenuation of the peak. Part of the storage is used for translation; that is, when the attenuated hydrograph is kinematic routed, some of the valley storage is filled. Because part of the valley storage does not cause attenuation, the storage volume used in the storage-routing portion of the Att-Kin procedure is related to the valley storage-discharge relationship by a constant of proportionality, C_2, as follows:

$$Q_{po} = k_s \, [C_2 S_{po}]^m \quad \text{--(14)}$$

and

$$C_2 = V_s/S_{po} \quad \text{--(15)}$$

V_s is determined by satisfying the conservation-of-mass equation at the time of the outflow hydrograph peak. It can be determined from the relationship:

$$V_s = V_0 - (V_t + V_d) \qquad (16)$$

Figure 2 illustrates these volumes. The storage volume (V_s) is that part of the valley storage which causes attenuation. The distortion volume (V_d) is a result of steepening of the rising limb of the storage routed

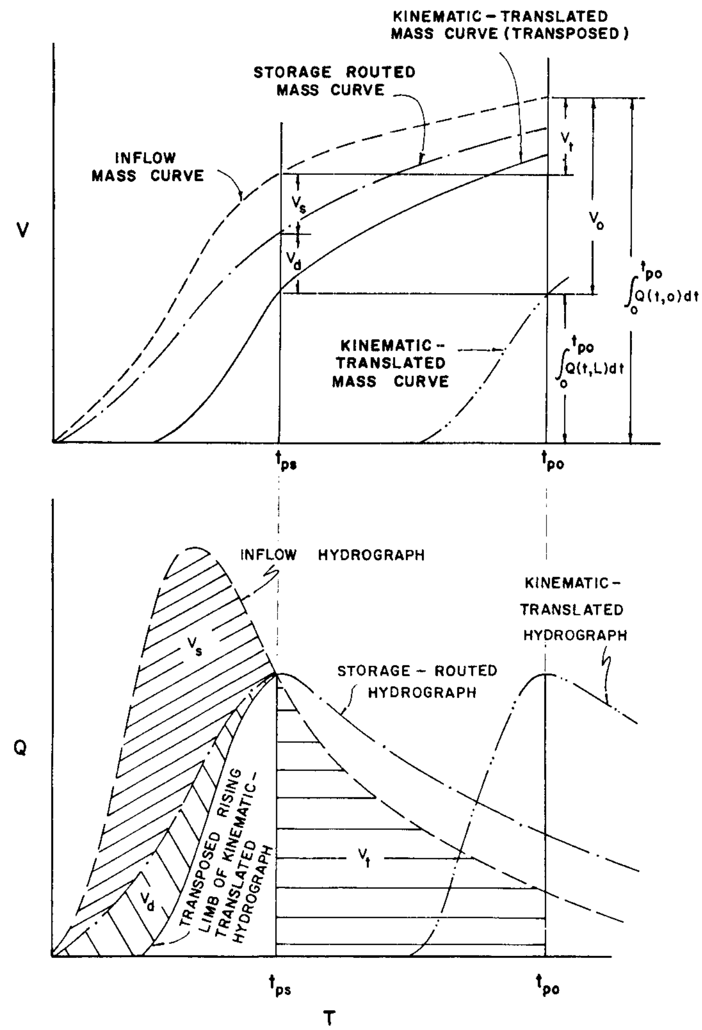

Figure 2. Volumes used in the Att-Kin routing procedure.

hydrograph. Offsetting the entire hydrograph by the kinematic travel time of the peak would not be correct. Each discharge travels at a different celerity resulting in a distortion of the rising limb of the hydrograph. This distortion is the difference between the travel times of the instantaneous and peak discharges. Integrating these differences over the rising limb of the hydrograph gives V_d. (This is illustrated by placing the kinematic-routed hydrograph so that its peak is coincident with the storage-routed hydrograph.) The volume V_t is the additional inflow that occurs between the times of the storage and kinematic-routed peaks. The maximum valley storage V_0 is the difference between the accumulated inflow volume and the accumulated outflow volume at the time to peak of the outflow hydrograph. Mathematically these volumes are

$$V_0 = \int_{t_o}^{t_{po}} (Q_{i,t} - Q_{o,t})dt \quad \text{------------------------------(17)}$$

$$V_t = \int_{t_{ps}}^{t_{po}} Q_{i,t} \; dt \quad \text{------------------------------------(18)}$$

$$V_d = L \int_{Q_b}^{Q_{po}} [(1/c) - (1/c_{po})]dQ \quad \text{-------------------------(19)}$$

The solution of the Att-Kin procedure is iterative until the V_s used in the storage routing balances the conservation of mass at the time of the outflow peak; i.e., V_s used in equation 15 results in a Q_{po} that gives the identical V_s in equation 16. At this point V_0 is equal to S_{po}.

The Modified Att-Kin Method

The modified Att-Kin method estimates the K that is to be used in equation 7. There is no iterative structure in the modified Att-Kin, but a direct solution is used. The K used approximates the slope of the storage-outflow curve at the peak outflow discharge. The best straight line estimate of the slope to use in equation 7 is the cord between the base flow and peak outflow points (see figure 3).

The storage used to compute K must be determined. This is the same storage V_s used in equation 15. The maximum valley storage, V_0, must be reduced by the V_d and V_t. Figure 4 illustrates the effect of reducing the storages, and shows the storage value (V_s) that is used.

The computation of the K-value is based on nondimensional units and reconverted to actual units for the routing. The nondimensional units used are:

$$Q_b^* = Q_b/Q_I \quad \text{---(20)}$$

$$Q_{po}^* = Q_{po}/Q_I \quad \text{--(21)}$$

$$Q^* = Q/Q_I = (Q_{po} - Q_b)/Q_I \quad \text{-------------------------------(22)}$$

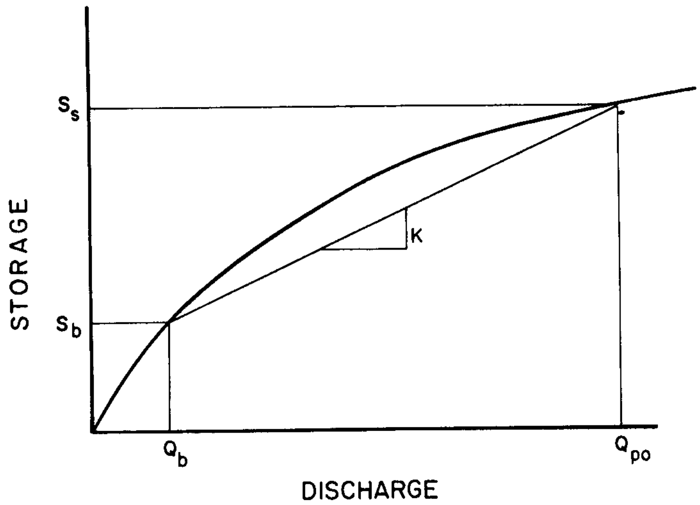

Figure 3. Storage-outflow slope used in the modified Att-Kin routing.

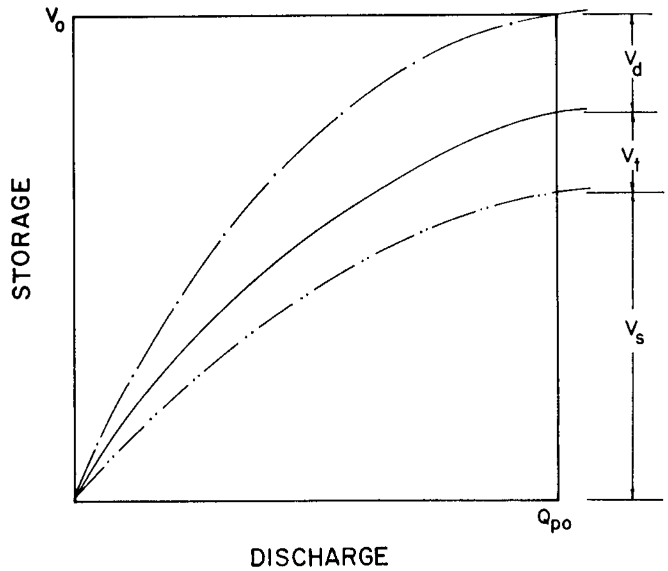

Figure 4. Storage value used in the modified Att-Kin routing.

$$S_b^* = V_b/V_I \qquad (23)$$

$$S_{po}^* = V_0/V_I \qquad (24)$$

$$S^* = (V_0 - V_b)/V_I \qquad (25)$$

$$S_s^* = V_s/V_I \quad \text{--}(26)$$

$$k^* = Q_I/(k_s\, V_I^{\,m}) = (S^*)^{\,m}/Q^* \quad \text{----------------------------}(27)$$

$$K^* = (S_s{}^* - S_b{}^*)/(Q_{po}^* - Q_b^*) \quad \text{--------------------------------}(28)$$

Since the actual outflow peak is not known, it must be estimated before the routing. The equation for Q^* is

$$Q^* = (S_{po}^*)^m/k^* - Q_b^* \quad \text{-------------------------------------}(29)$$

Since S_{po}^* contains S_b^*, S_{po}^* can be written as

$$S_{po}^* = S^* + S_b^* \quad \text{---}(30)$$

Combining equations 29, 30, and a nondimensional form of equation 9 and assuming $S^* = 1$ yields

$$\hat{Q}^* = \{(1/k^*)^{1/m} + Q_b^{*\ 1/m}\}^m - Q_b^* \quad \text{---------------------------}(31)$$

where $\hat{Q}^*$ is an estimate of Q^*.

To limit the $\hat{Q}^*$ to 1 or less,

$$\hat{Q}^* = \min\ \{([(1/k^*)^{1/m} + Q_b^{*1/m}]^m - Q_b^*)\ ,1\} \quad \text{-----------------}(32)$$

Equation 19 can be integrated directly to yield

$$V_d = V_0\ \{1 - 1/m + 1/m\ (Q_b/Q_{po})\} - V_b \quad \text{---------------------}(33)$$

or, in nondimensional form,

$$V_d^* = S_{po}^*\ \{1 - 1/m + 1/m\ (Q_b^*/Q_{po}^*)\} - S_b^* \quad \text{--------------------}(34)$$

Because V_t cannot be calculated without integrating part of the inflow hydrograph, V_t is not used in this procedure. To compensate for not using it, a curve fit was used to modify V_d so that a close approximation to the data was achieved.

Multiplying the V_d by $\hat{Q}^{*\frac{1}{2}}$ and substituting the result for $(V_d + V_t)$ gave an excellent fit to the simplified Att-Kin procedure in TR-66. The overcorrection of V_d compensates for the lack of the V_t correction.

Simplifying and combining equations 16, 28, 34, and setting $(V_d + V_t) = V_d\hat{Q}^{*\frac{1}{2}}$ yield the following:

$$K^* = \{k^*\ (\hat{Q}^* + Q_b^*)^{1/m}/\hat{Q}^*\}\{(1 - \hat{Q}^{*\frac{1}{2}})[1 - \{Q_b^*/(\hat{Q}^* + Q_b^*)\}^{1/m}]$$
$$+ (\hat{Q}^{*\frac{1}{2}}/m)\ [1 - \{Q_b^*/(\hat{Q}^* + Q_b^*)\}]\} \quad \text{-----------------------}(35)$$

where $\hat{Q}^*$ is defined by equation 32.

For computations, K is calculated as

$$K = (V_I/Q_I)\ K^* \quad \text{--} (36)$$

Using the K from equation 36, C_R is computed by equation 7 but not allowed to exceed 1. The routing is done using equation 6.

The resulting outflow hydrograph is then positioned in time so the difference between the time to peak of inflow and outflow is

$$\Delta t_p = \{S_{po}/(Q+Q_b)\}[\{(Q_I + Q_b)/(Q + Q_b)\}^{1/m} - 1]$$
$$\div [\{(Q_I + Q_b)/(Q + Q_b)\} - 1] \quad \text{-------------------------} (37)$$

The equation for Δt_p is developed from the kinematic travel time and assumes that peak discharge varies linearly with time during its movement through the reach.

The time to the peak of the storage-routed hydrograph (Δt_{ps}) is used to position the outflow hydrograph if $\Delta t_{ps} > \Delta t_p$.

VERIFICATION

Results from the modified Att-Kin procedure were compared with the solution of the complete unsteady flow equations. The method described by Theurer et al. (1976) was used for solving the complete equations. The computations were made for a hypothetical rectangular channel 3,048 m long and 3.05 m wide with 0.001 slope and mannings n-value of 0.025. The inflow hydrographs were gamma functions of varying properties.

The results are compared by plotting Q^* vs. k^*. The parameter k^* reflects the properties of the inflow hydrograph and the channel, and it is a proper nondimensional basis to use for comparisons (Theurer and Comer, 1979). Upper and lower limits bracket the solutions, and any unsteady flow solution must fall within these limits to be valid. The lower limit is a linear storage model and is valid as long as base flow is not a significant part of the hydrograph. The upper limit consists of two parts: The kinematic model and kinematic model plus the conservation of mass. The kinematic model is a valid upper bound as long as it accounts for kinematic shock. When shock reaches the peak of the inflow, then the conservation of mass must be included also as the upper limit.

Two tests illustrate the performance of the Att-Kin procedure. The first test routed an inflow hydrograph with the following properties: base flow, 2.83 m^3/s; peak flow, 28.3 m^3/s (25.5 m^3/s above base flow); time to peak, 15 minutes; and time to center of mass, 20 minutes.

The results of this test are shown in figure 5. The Att-Kin performs well in this situation; i.e., differences between the complete unsteady flow solution and the Att-Kin are small. The "hook" on the end of the complete unsteady flow solution is caused by the critical flow condition at the lower end of the channel, where flow is so close to critical that there is almost pure translation of the peak.

The second test was the routing of an inflow hydrograph with the following properties: base flow, 28.3 m^3/s; peak flow, 56.6 m^3/s (28.3 m^3/s above base flow); time to peak, 1,000 seconds; time to center of mass, 2,000 seconds. Figure 6 shows the results. In this case the lower limit

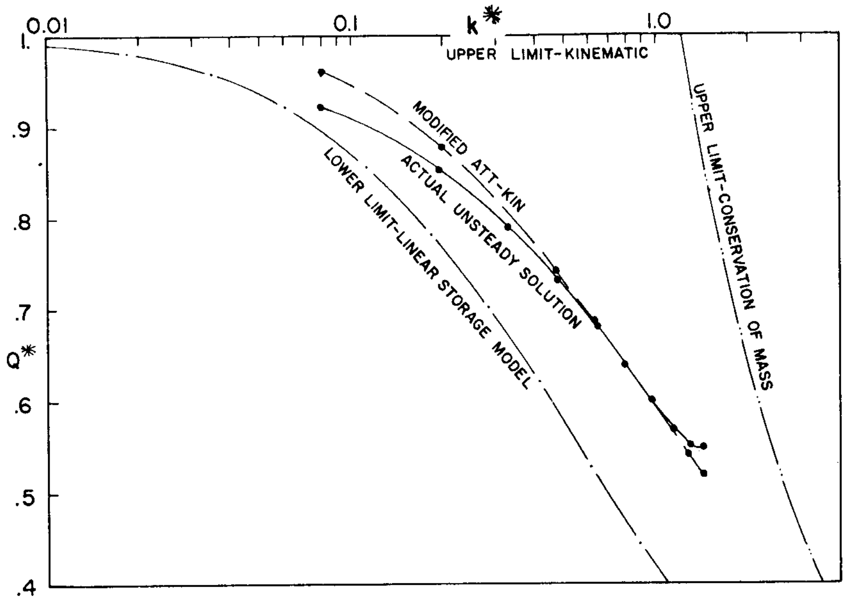

Figure 5. Q* vs. k* for low base flow.

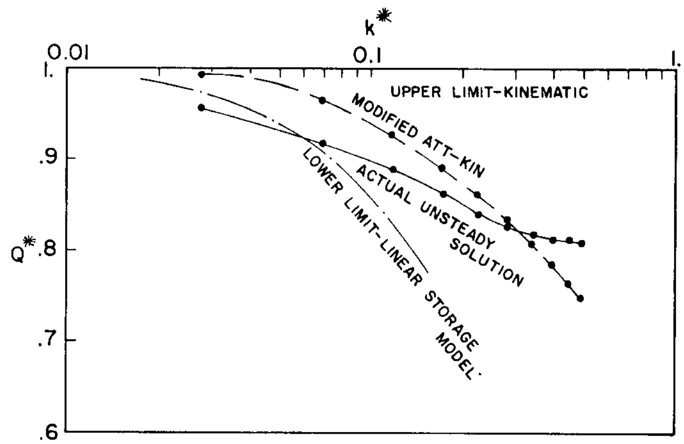

Figure 6. Q* vs. k* for high base flow.

is not valid because the valley storage used for translation is large compared to that causing attenuation. The "hook" on the complete unsteady solution is also more pronounced in this situation because the influence of critical depth is carried farther upstream.

DISCUSSION AND SUMMARY

The modified Att-Kin routing model was developed by analysis of the physical situation and was modified to fit the existing structure of the SCS TR-20 computer program. The current algorithm does not have provision for lateral inflows or outflows, but the TR-20 computer program does provide for adding inflow hydrographs at either end of a channel reach. The modified Att-Kin model was verified by comparing the test results to the results of the complete dynamic equations.

Although the tests were made using a rectangular channel, there are no inherant limits on the channel shape that can be used. The two tests used extremes of base flow and of rapidly rising hydrographs that are typical of situations encountered in upland watersheds. The performance of the modified Att-Kin model in these tests indicates that the method will give good predictions of the downstream peaks for the conditions of concern to SCS. The model should perform well in situations where a single valued rating curve adequately represents the momentum equation.

The model has been put into the TR-20 computer program, and it is being tested on a wide range of field problems.

LIST OF SYMBOLS

A = valley cross-section area

A_{po} = area associated with the peak of the outflow

$A_{x,t}$ = area at distance x and time t

$C = \Delta t/K$; convex routing coefficient

$C_R = (2\Delta t)/(2K + \Delta t)$; modified Att-Kin routing coefficient

$c = m\,(Q/A)$; celerity of the wave

$c_{po} = m(Q_{po}/A_{po})$; celerity of the wave associated with the peak discharge

I = inflow

K = slope of storage outflow curve

k_a and m_a = coefficient and exponent, respectively, for the single-valued area-discharge relationship

k_s and m = coefficient and exponent respectively for the single-valued storage-discharge relationship

L = valley length

O = outflow

Q = outflow hydrograph peak above base flow

Q_b = base flow discharge

Q_I = inflow hydrograph peak above base flow

$\bar{Q}_I$ = average inflow over the time interval $t_2 - t_1$

$\bar{Q}_o$ = average outflow over the time interval $t_2 - t_1$

Q_{po} = the peak of the outflow hydrograph

$Q_{i,t}$ = inflow discharge

$Q_{o,t}$ = outflow discharge

$Q_{x,t}$ = discharge at distance x and time t

S = valley storage

S_{po} = maximum valley storage in the reach during the passage of and assumed coincident with the outflow peak

$S_{o,t}$ = valley storage in the reach at time t

t = time

t_1 = time at the beginning of the interval

t_2 = time at the end of the interval

t_{po} = time to peak of the outflow (kinematic-routed) hydrograph

t_{ps} = time to peak of the storage-routed hydrograph

Δt_p = elapsed time between the inflow and outflow peaks

Δt_{ps} = time to peak of storage routed hydrograph

V_b = volume of base flow

V_d = volume of distortion due to kinematic routing

V_I = volume of inflow above base flow

V_O = volume difference between the volumes of inflow and outflow at the time to peak of the outflow hydrograph

V_s = net storage used in computing the outflow peak by the storage routing submodel.

V_t = inflow volume between the times of the storage and of the storage kinematic-routing peaks

x = distance along the channel

REFERENCES

1. Soil Conservation Service. 1965. Computer program for project formulation--hydrology. Tech. Rel. 20. 28 p., exhibits. Suppl. 1, 1969.
2. Soil Conservation Service. 1972. National engineering handbook. Section 4. Hydrology.
3. Soil Conservation Service. 1979. Simplified dam-breach routing procedure. Tech. Rel. 66. 36 p.
4. Theurer, Fred D., A. H. Barnes, and E. V. Richardson. 1976. A solution for unsteady open channel flow. Rivers 76. Vol. II American Society of Civil Engineers. p. 1497-1516.
5. Theurer, Fred D., and George H. Comer. 1979. Verification of SCS dam-breach routing procedure. Paper No. 79-2577 Presented at 1979 winter meeting of American Society of Agricultural Engineers. December 11-14, 1979, New Orleans, Louisiana. 22 p.